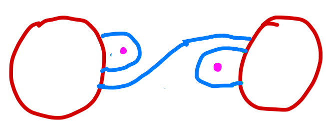
D Heegaard diagrams and knot

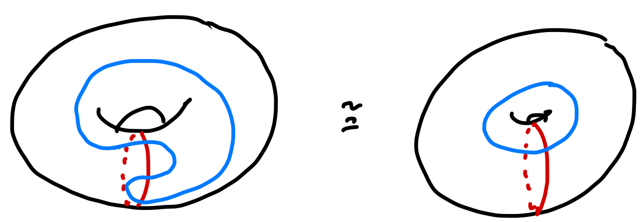
there are several ways to represent a knot K in a Heegaard diagram

1. Doubly pointed diagram
2. Diagram $(\Sigma_g, \alpha_1, \dots \alpha_g, \beta_1, \dots \beta_{g-1})$ for Y-nbhd(K)

① consider



this is a knot is S^3

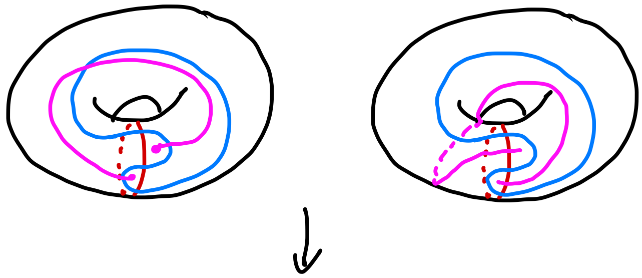


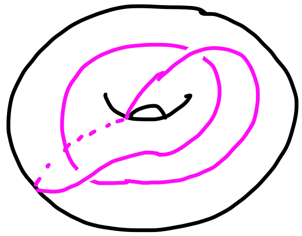
S^3

now ① take an arc α_1 on Σ connecting points and disjoint from the red
push its interior below Σ
(i.e. into red handlebody)

② take an arc α_2 on Σ connecting points and disjoint from the blue
push its interior above Σ
(i.e. into blue handlebody)

$\alpha_1 \cup \alpha_2$ a knot!





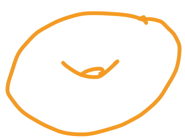
left handed trefoil!

lemma 6:

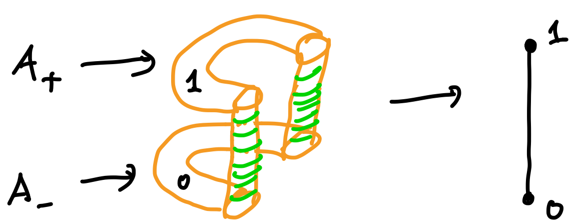
> any knot $K \subset Y$ can be represented by a doubly pointed Heegaard diagram for Y

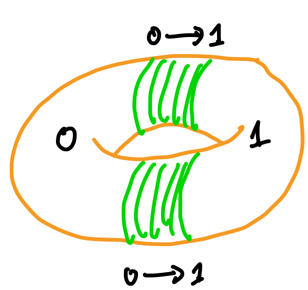
Proof: given $K \subset Y$

let $Y_K = \overline{Y - N(K)}$ where $N(K)$ is a nbhd of K

$\partial Y_K =$  T^2

define a function $f : T^2 \to \mathbb{R}$ by





now extend f to the interior of Y_K so

the interior maps to $(0,1)$

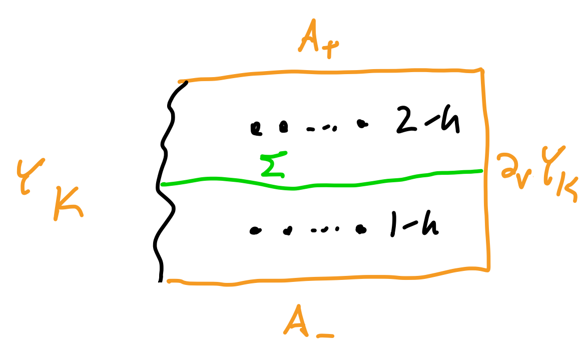
approximate f by a Morse function (without changing on the boundary

so we are thinking of Y_K as a cobordism of manifolds with boundary

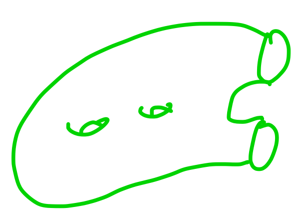
$$\partial_\pm Y_K = A_\pm \qquad \partial_v Y_K = 2 \text{ annuli}$$

we know we can assume Y_K has a handlebody structure with <u>no</u> 0 or 3-handles

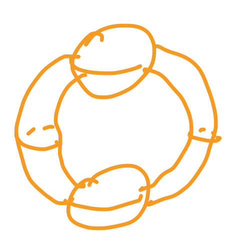
and all 1-handles come before 2-handles



so the "Heegaard" surface is a surface of genus g with 2 boundary compts



now lets put back $N(K)$

note: $N(K) =$ 

2 0-handles
2 1-handles

we write it this way because we think of ∂_+ of 1-handles as $\partial_v Y_K$

note: they are attached to Y_K along $\partial_v Y_K = 2$ annuli

so they are attached to Y_K as 2-handles

when we attach these to Y_K we change Y_K from

$$(\Sigma \times [0,1]) \cup (\alpha_i \text{ 2-handles attached to } \Sigma \times \{0\})$$
$$\cup (\beta_i \text{ ——— } \Sigma \times \{1\})$$

to

$$Y' = \bar{\Sigma} \times [0,1] \cup (\alpha_i \text{ 2-handles attached to } \Sigma \times \{0\})$$
$$\cup (\beta_i \text{ ——— } \Sigma \times \{1\})$$

where $\bar{\Sigma} = \Sigma \cup 2$ disks to cap off boundary

note: $Y' = Y - 2(\text{3-balls})$

and we get Y from Y' by gluing in 0-handles from $N(K)$

one of these is a 0-handle for Y

the other is a 3-handle

now, where is K?

K is the core of $N(K)$ so it intersects $\bar{\Sigma}$ in 2 points we can push the part of K in the upper handlebody onto Σ so it is disjoint from β_i's

similarly for the part in the lower handle body

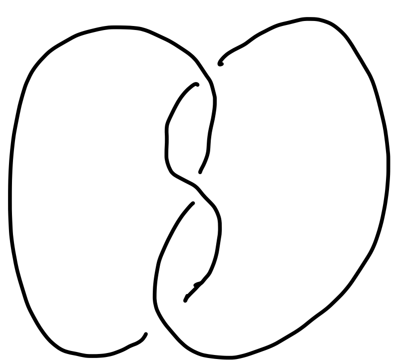
this is exactly the description of K in the construction above!

note: by construction K = union of gradient flowlines through the 2 points!

We know the doubly pointed diagrams exist, but how to find them for knots in S^3?

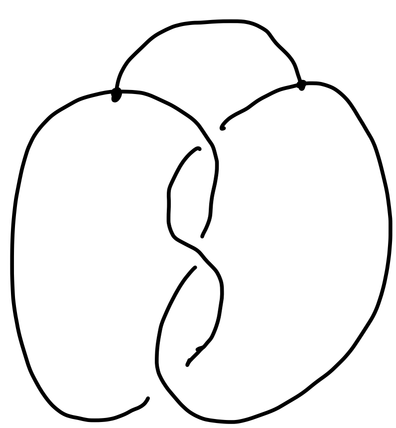
We start with an example:

consider



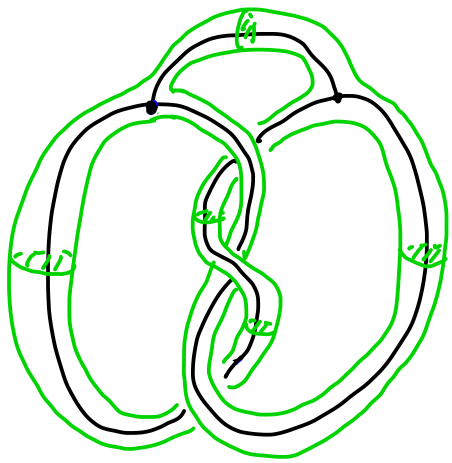
note a neighborhood of this is a handlebody H

(nbhd of any graph is!)



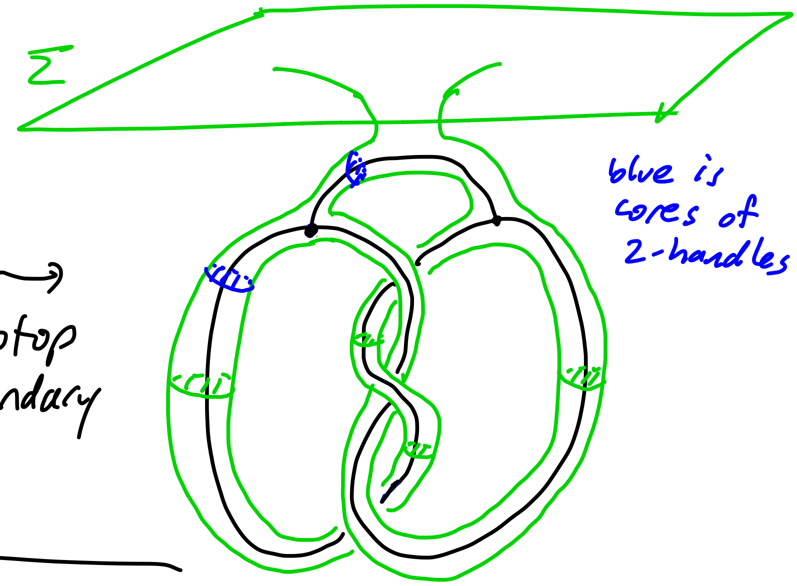
$H =$

$\Sigma = \partial H$



isotop boundary

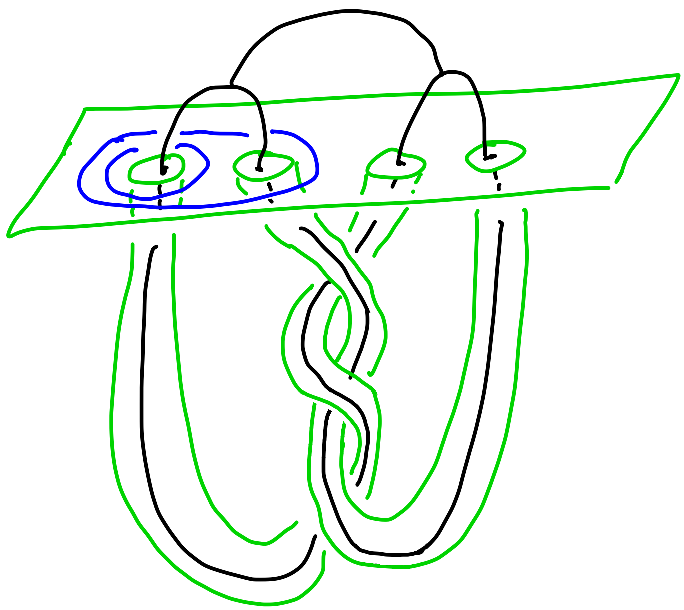
Σ



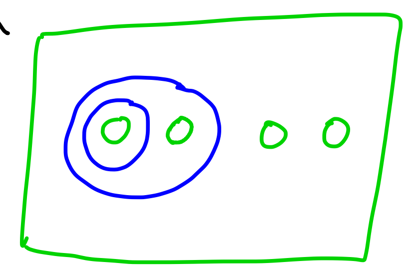
blue is cores of 2-handles

isotop boundary

Σ



surface of genus 2



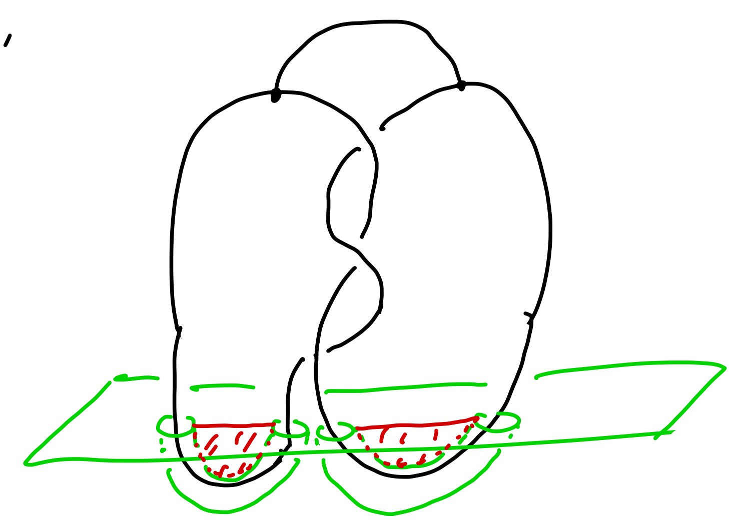
attaching spheres of β 2-handles describes H

Claim: $\overline{S^3 - H}$ is a handlebody H'

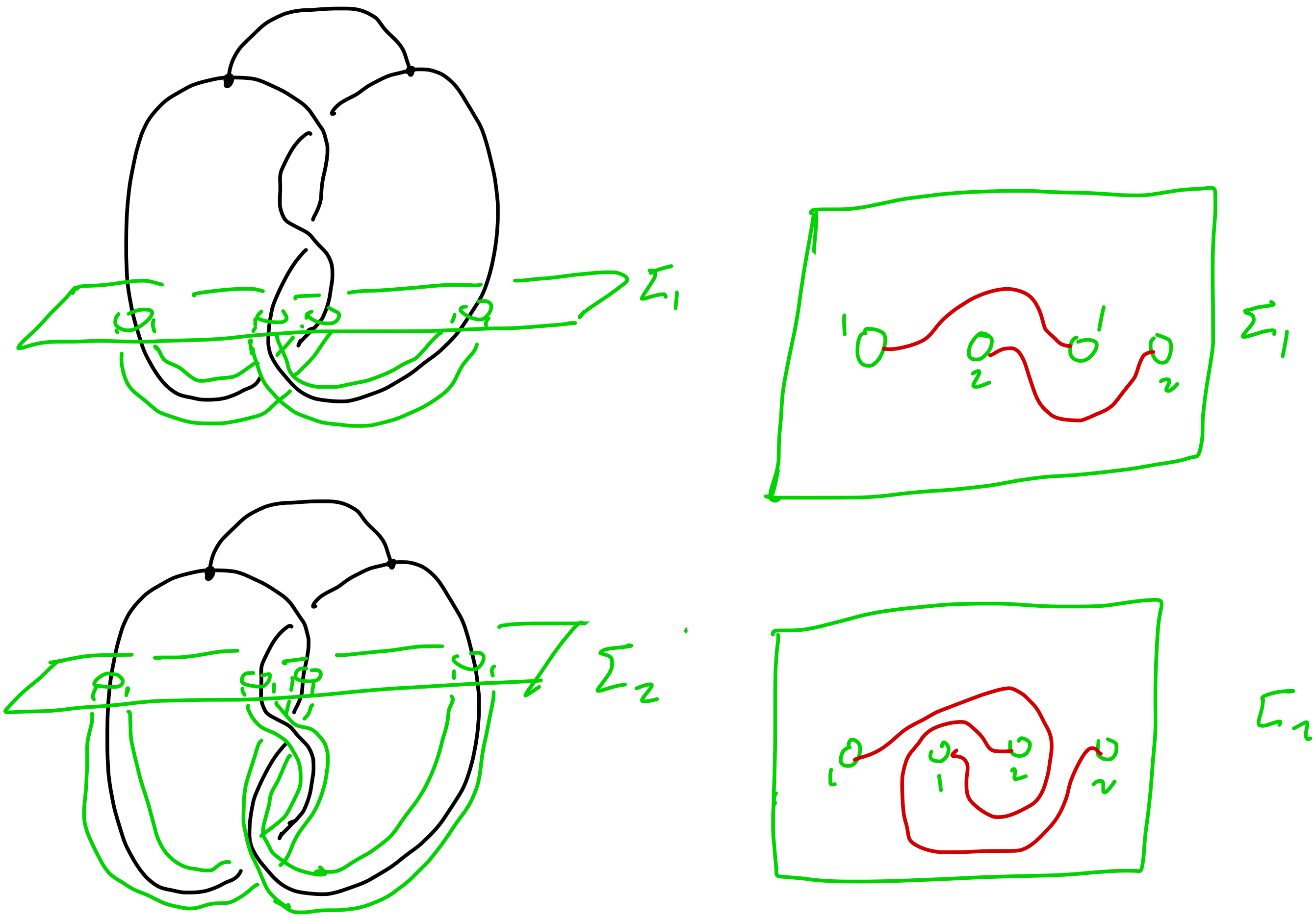
note Σ is isotopic to Σ'

the red show H' is a handle body

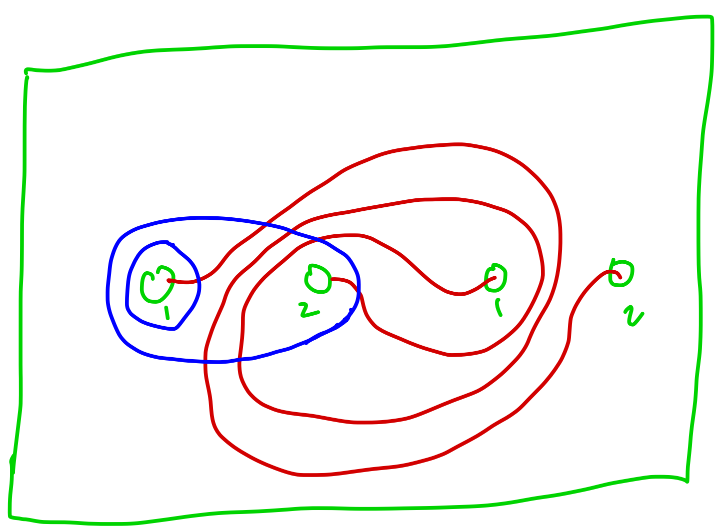
Σ'



track the red curves as you push Σ' up to Σ



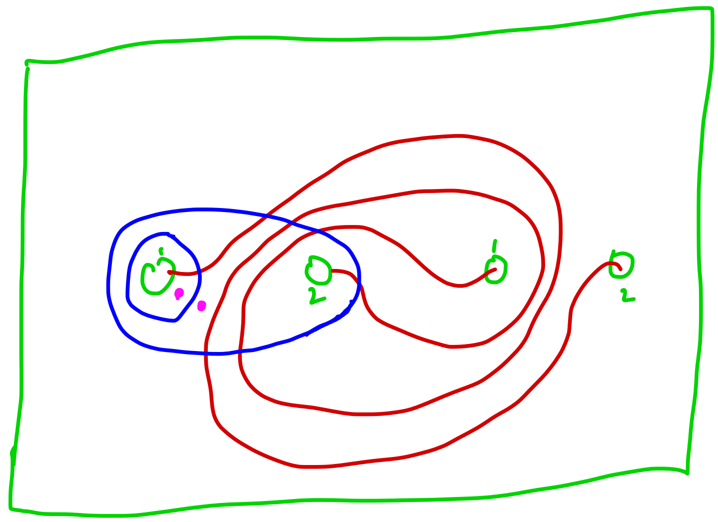
so on Σ we see



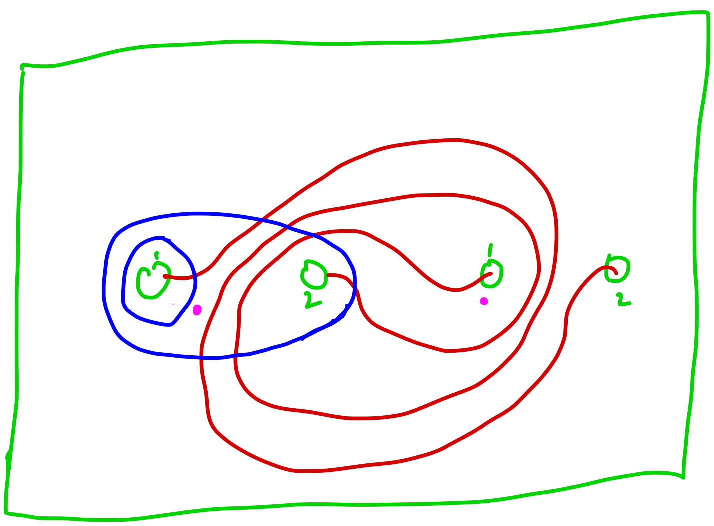
this describes a Heegaard splitting of S^3

<u>exercise</u>: Check this.

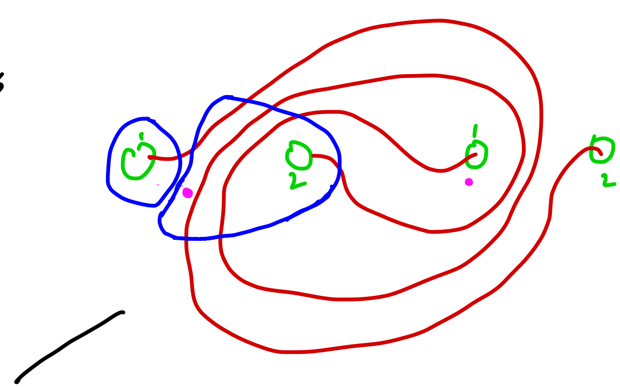
<u>exercise</u>: Show the 2 points below describe K



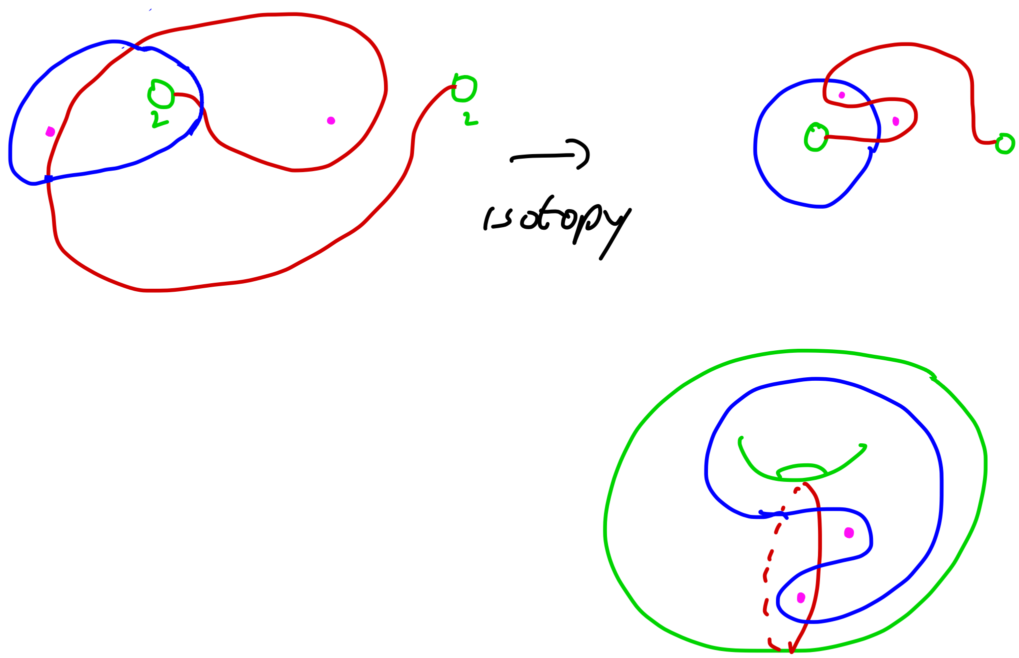
<u>note</u>:



we can now slide handles

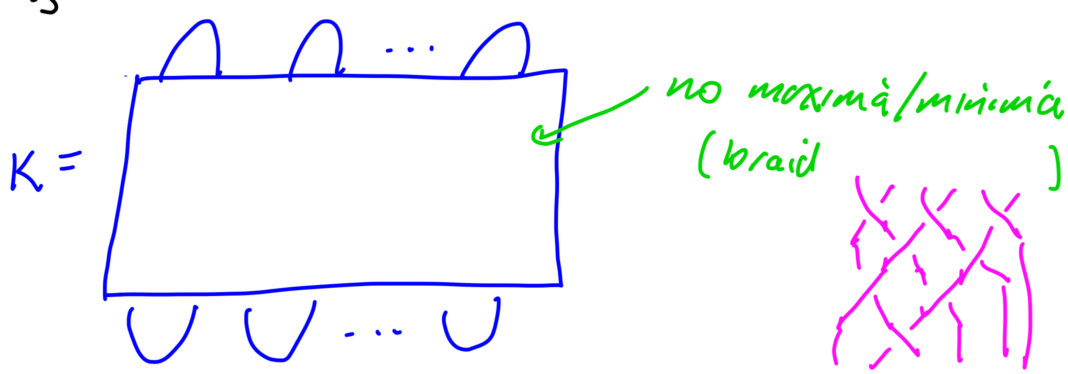


↙ isotopy

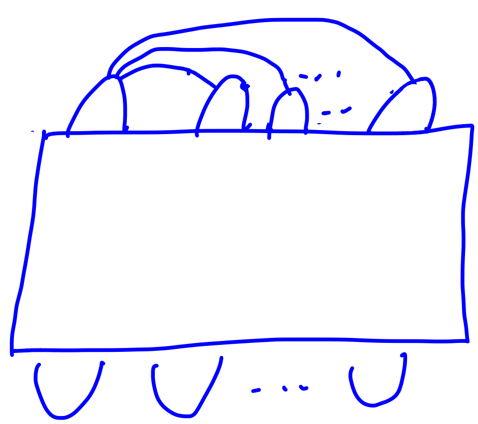


in general, any knot can be put in "bridge position"

that is



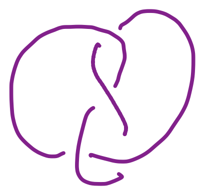
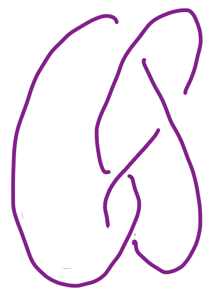
then a nbhd of

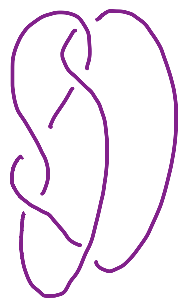


is a handlebody and one can use above procedure to build doubly pointed diagram for K

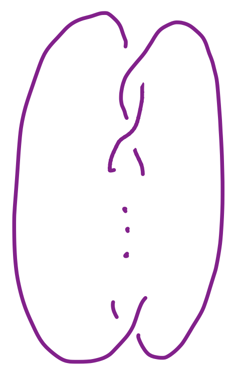
exercise:

compute doubly pointed diagram for

1)  $\simeq$ 

2) 

3) $(2, 2n+1)$ torus knot

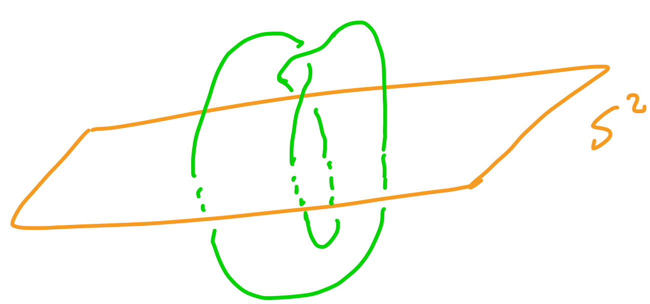


$2n+1$ twists

Hint: all fit on a genus 1 Heegaard diagram

4) try a 3-bridge knot

5) any knot $K \subset S^3$ of the form



i.e. n disjoint arcs α_i on S^2

" " β_i on S^2

s.t. $K = \bigcup (\alpha_i$ with interior pushed down)
$\cup \bigcup (\beta_i$ -- " up)

(another way to say K is an n-bridge knot)

so we have a "$2n$ pointed" diagram for K

Show how to stabilize Heegaard diagram to get a 2-pointed diagram for K.

2. 2^{nd} way to describe a knot is by a Heegaard diagram for $Y_K = \overline{Y - \text{nbhd}(K)}$

note: $\partial Y_K = T^2$ so we can take a Morse function

$$f : Y_K \to [0,1]$$

s.t. $f^{-1}(1) = T^2$

from this we get a Handlebody and we can assume there are no 3-handles and just one 0-handle

thus Y_K is described by a Heegaard diagram

$$(\Sigma_g, \{\alpha_1, \dots \alpha_g\}, \{\beta_1, \dots, \beta_{g-1}\})$$

moreover there is a curve β_g on ∂Y_K (and therefore on Σ_g disjoint from $\beta_1 -- \beta_{g-1}$ that corresponds to the meridian of K

if you attach a 2-handle to β_g too

you get Y

also note there is a curve λ on ∂Y_K and

$\therefore$ on Σ_g disjoint from $\beta_1, \dots, \beta_{g-1}$ but

intersecting β_g one time

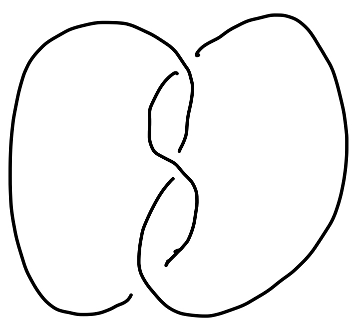
that corresponds to a framing curve

for K

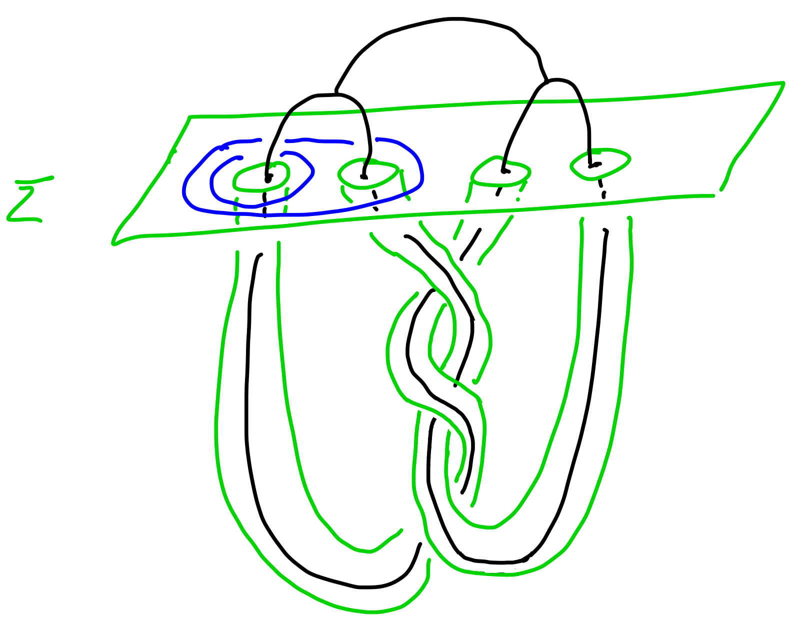
example: we can use the procedure above to find

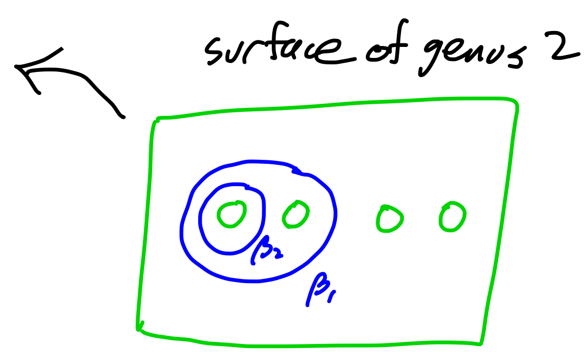
such a Heegaard diagram

recall for

$K =$ 

we had the Heegaard surface Σ

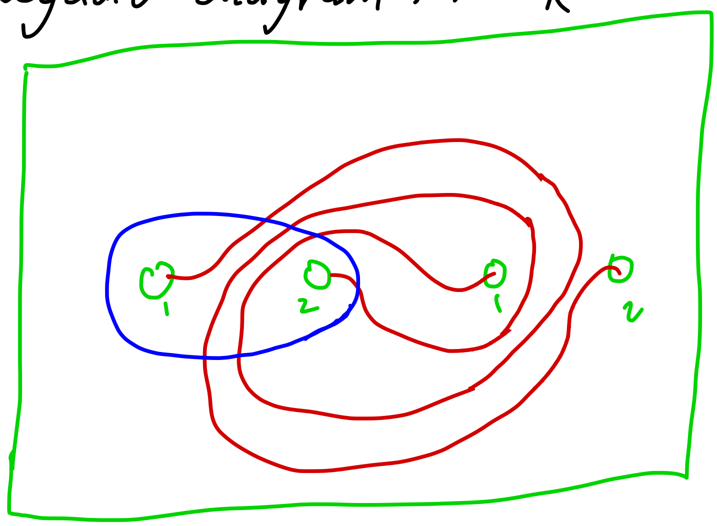




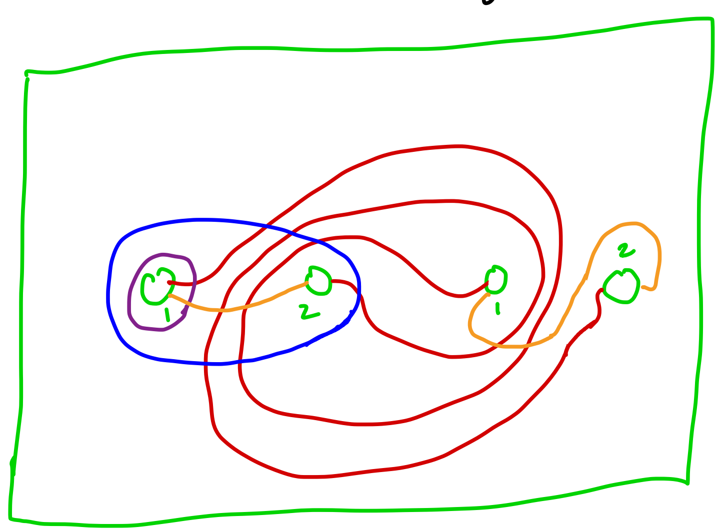
attaching spheres of β 2-handles describes H

note: β_2 is a meridian for K

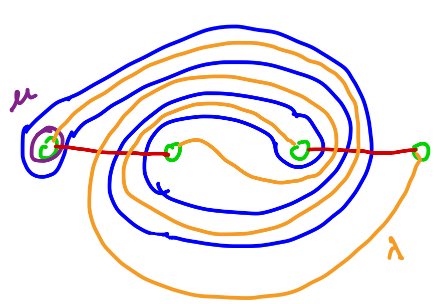
so a Heegaard diagram for S^3_K is



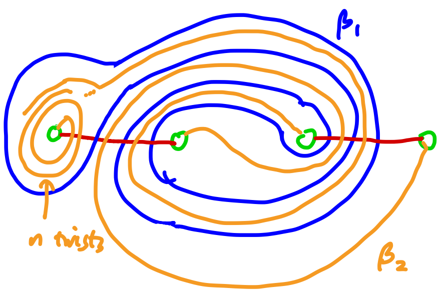
and the meridian and a longitude are



exercise: 1) show that for the right handed trefoil you get



2) show k-framed surgery on the right handed trefoil is



for some n, determine n in terms of k

3) use Heegaard diagram above to see

6 surgery on right handed trefoil is $L(3,1) \# L(2,1)$

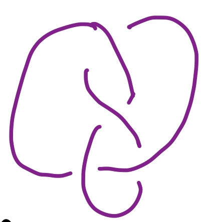
5 ———————————— $L(5,1)$

7 ———————————— $L(7,3)$

0 ———————————— a T^2-bundle over S^1

exercise:

1) draw a diagram for the complement of the figure 8 knot



2) show ± 3 surgery on the figure 8 knot contains an incompressible torus

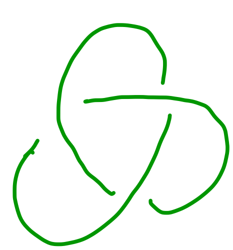
± 4 ————————————————————

0 ———————————— is a T^2 bundle over S^1

Hint: I don't know how hard these are!

Here is another way to get a Heegaard diagram for Y_K when $Y = S^3$

start with a knot diagram:

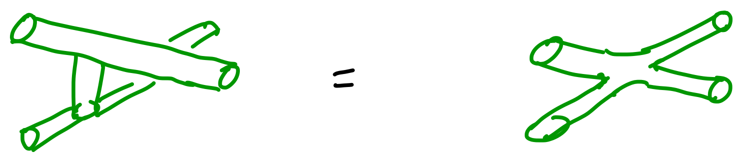


create a graph G that is $K \cup$ arcs at crossings

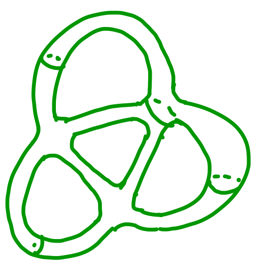
let H = nbhd of G clearly a handlebody

<u>Claim:</u> $\overline{S^3 - H}$ a handlebody

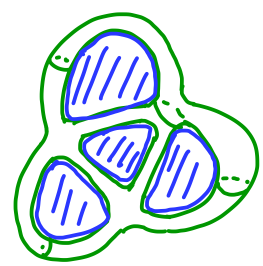
note



so H is



and we have disk in complement

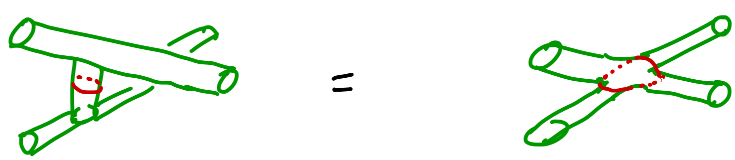


cutting complement of H along disks gives B^3

$\therefore S^3 - H$ a handlebody and we know compressing disks (i.e. 6 curves)

what are compressing disks for H?

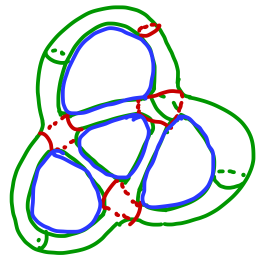
at each crossing have



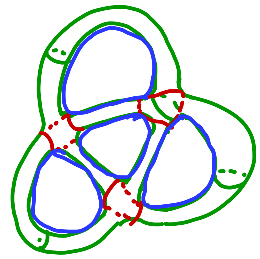
compressing along these gives $S^1 \times D^2$

now compress along meridian to get B^3

so we have for S^3

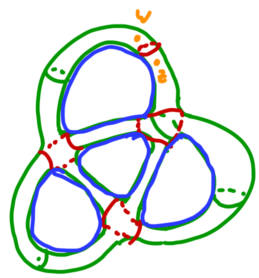


and for S^3-nbhd(K)



just remove the meridian

a doubly pointed diagram for K is



exercise: 1) find the longitude on ∂S^3_K

2) find Heegaard diagrams various surgeries on K and compair to diagrams created above

3) Simplify this diagram to the one created above.

exercise:

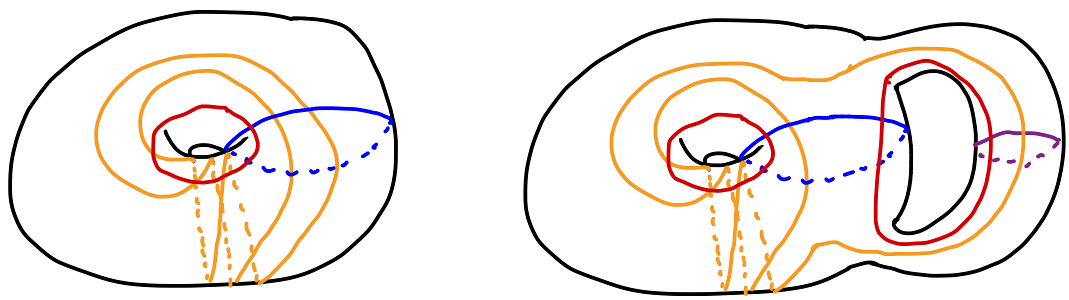
1) given a doubly pointed diagram for K find a diagram for the complement of K

Hint: Stabilize

2) go the other way, i.e. given a diagram for Y_K draw a doubly pointed diagram for Y_K

3) if K sits on a Heegaard diagram for Y show how to find a diagram for Y_K

e.g. for the right handed trefoil we have



E. Fibered knots

(this section is based on work of Gabai)

We would like to understand the following:

given
- 3-manifold Y
- a link $L \subset Y$
- a connected surface $\Sigma \subset Y$ s.t. $\partial \Sigma = L$

is there a fibration

$$\pi : (Y - L) \to S^1$$

s.t. Σ is a fiber ?

recall, a surtured 3-manifold is a cobordism M of surfaces with boundary such that $\partial_v M$ is a product $\partial_- M \times [0,1]$

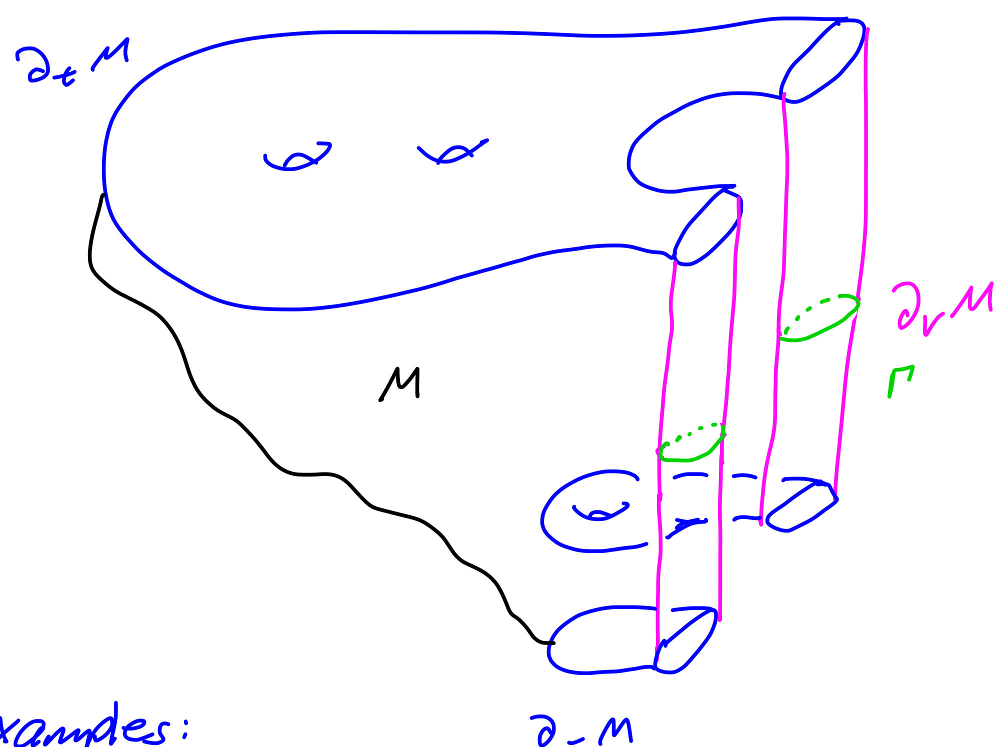
note: $\partial_v M$ is a union of annuli call

let $\Gamma = \bigcup$ core circles in the annuli

the circles are called <u>sutures</u>

so $\partial_v M = $ nbhd of Γ

Γ determines $\partial_v M$ and mostly $\partial_\pm M$ so we denote the sutured manifold by (M, Γ)



examples:

let Σ be a surface with boundary

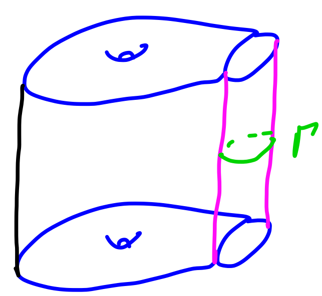
$M = \Sigma \times [0,1]$ is a sutured manifold

$\partial_+ M = \Sigma \times \{1\}$

$\partial_- M = \Sigma \times \{0\}$

$\partial_v M = \partial\Sigma \times [0,1]$

$\Gamma = \partial\Sigma \times \{1/2\}$



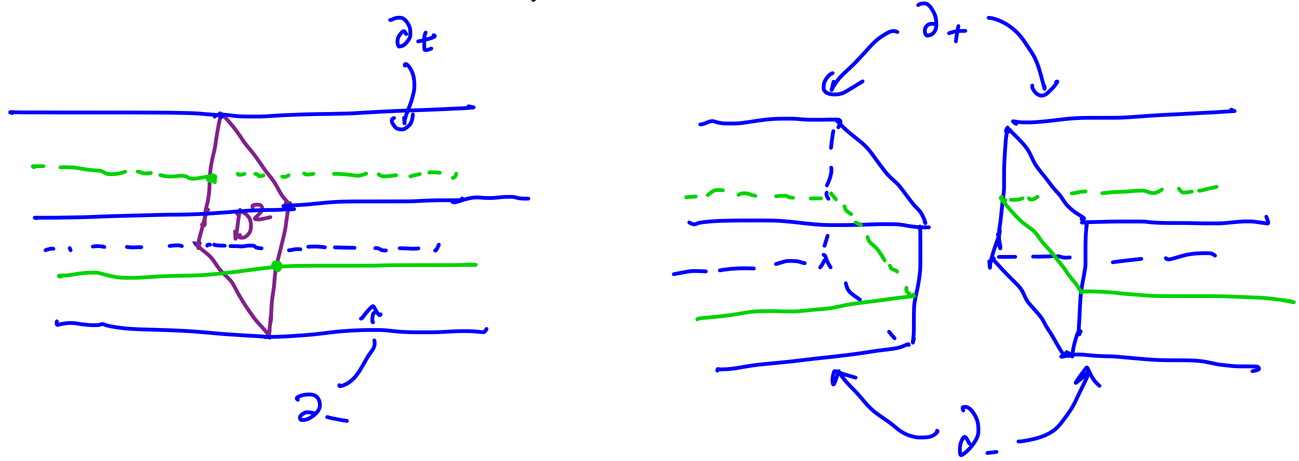
this is called a <u>product sutured manifold</u>

we call a properly embedded disk $D^2 \subset (M, \Gamma)$

a product disk if ∂D^2 intersects Γ transversely

and $\partial D^2 \cap \Gamma$ = 2 points

note: given such a disk we get a new sutured manifold (M', Γ')



note: in $\partial M'$ there are 2 copies of D^2, call them $D_\pm$ and $D_\pm \cap \Gamma$ = arc

write $(M, \Gamma) \overset{D^2}{\rightsquigarrow} (M', \Gamma')$

lemma 7:

if $(M, \Gamma) \overset{D^2}{\rightsquigarrow} (M', \Gamma')$ and D^2 is a product disk

then (M, γ) a product sutured manifold

$\Leftrightarrow$

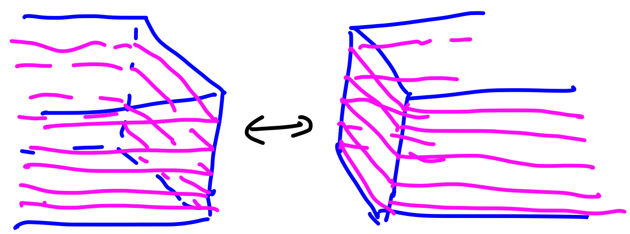
(M', γ') a product sutured manifold

Proof: $(\Leftarrow)$ if $M' = \Sigma \times [0,1]$, note $D_\pm \subset \partial_v M = \partial\Sigma \times [0,1]$

and (after isotopy) can assume $D_\pm = I_\pm \times [0,1]$

where I_+, I_- are disjoint intervals

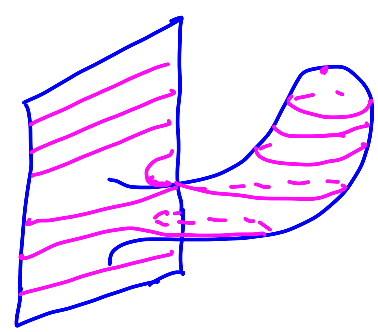
get M from $\dot{M}$ by gluing D_+ to D_- and we can do this preserving the product str!



($\Longrightarrow$) if $M = \Sigma \times [0,1]$, then a product disk can be isotoped so $D^2 = c \times [0,1]$ where c an arc in Σ

to see this note we can make D transverse to $\Sigma \times \{t\}$ for all but finitely many t and at these t have a critical point of projection $\Sigma \times [0,1] \to [0,1]$ restricted to D

eg.



exercise: show  bounds a ball (i.e. $\Sigma \times [0,1]$ is irreducible)

use ball to isotop to remove 2 critical

points

when there are no critical points easy to isotop D to $C \times [0,1]$ as claimed

now, given $D = C \times [0,1]$ we see

$$M' = (\Sigma \setminus C) \times [0,1]$$

above we mentioned irreducible

a 3-manifold M is irreducible if any embedded sphere $S^2 \subset M$ bounds a ball in M

exercise:

1) $D \subset M$ properly embedded disk

then M irreducible

$\Longleftrightarrow$

$M \setminus D$ irreducible

2) $\Sigma \times [0,1]$ is irreducible

3) knot complements in S^3 are irreducible

lemma 8:

if (M, Γ) irreducible, $\partial M = S^3$, and Γ connected then $M \cong D^2 \times [0,1]$ and Γ isotopic to $\partial D^2 \times \{1/2\}$

Proof: irreducible $\Rightarrow M \cong B^3 \cong D^2 \times [0,1]$

and any 2 curves on $S^2 = \partial B^3$ are isotopic

Th$^{\underline{m}}$ 9:

(M,Γ) an irreducible sutured manifold

(M,Γ) is a product sutured manifold

$\Longleftrightarrow$

$\exists$ a sequence

$$(M,\Gamma) \overset{D_1}{\rightsquigarrow} (M_1,\Gamma_1) \overset{D_2}{\rightsquigarrow} \cdots \overset{D_k}{\rightsquigarrow} (M_k,\Gamma_k)$$

where each D_i is a product disk and
∂M_k = union of S^2s and Γ_k = one circle in each S^2

Proof: clear from lemmas 7 & 8 and exercise that M irreducible $\Leftrightarrow$ $M \setminus D^2$ irreducible ∎

we say (M,Γ) is disk decomposable if $\exists$ a sequence of product disks in (M,Γ) that cut (M,Γ) to (M_k,Γ_k) as in theorem so th$^{\underline{m}}$ says (M,Γ) disk decomposable iff (M,Γ) a product sutured mfd.

now, given $L \subset Y$ and $\Sigma \subset Y$ a surface with $\partial\Sigma = Y$

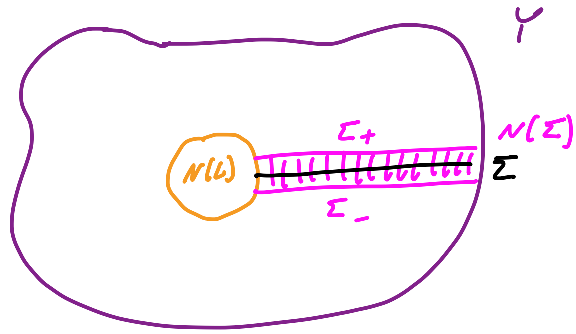
we can consider $Y_L = Y - N(L)$ $\qquad N(L)$ = nbhd L

and $\Sigma \subset Y_L$ (actually $\Sigma \cap Y_L \cong \Sigma$)

$Y-L$ fibered $\Leftrightarrow$ Y_L fibered

note: a nbhd of $\Sigma \subset Y_L$ is $N(\Sigma) = \Sigma \times (-\varepsilon,\varepsilon)$

let $Y_\Sigma = Y_L - N(\Sigma)$



$$\partial N(\Sigma) = -\Sigma_- \cup \Sigma_+ \cup (\partial\Sigma \times (-\varepsilon,\varepsilon))$$

$$\text{so } \partial Y_\Sigma = \Sigma_- \cup -\Sigma_+ \cup \underbrace{\big(\partial N(L) - (\partial\Sigma \times (-\varepsilon,\varepsilon))\big)}_{\text{annulus for each component of } L}$$

so Y_Σ is a sutured manifold

with sutures Γ a circle in each annulus of $\partial N(L) - (\partial\Sigma \times (-\varepsilon,\varepsilon))$

Thm 10:

$$\boxed{L \text{ is fibered in } Y \text{ with } \Sigma \text{ being a fiber} \iff Y_\Sigma \text{ is disk decomposable}}$$

Proof: almost imediate from above

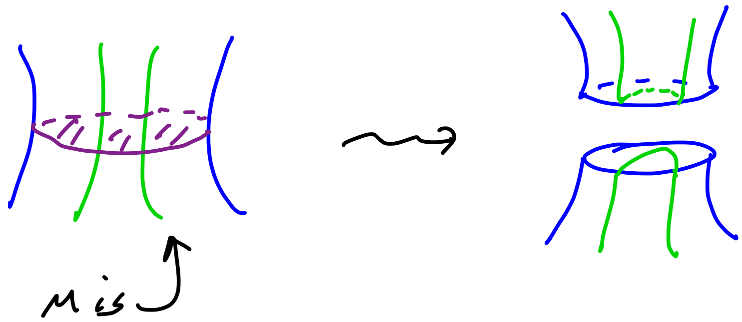
exercise ▦

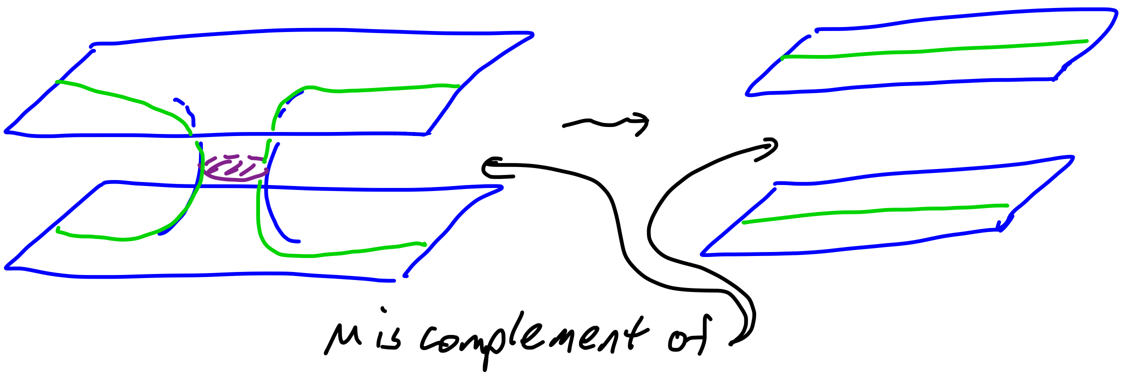
we say $L \subset Y$ is disk decomposable if Y_Σ is

now we want to determine if a link in S^3 is fibered

so we need to see if the complement of a surface in S^3 is disk decomposable

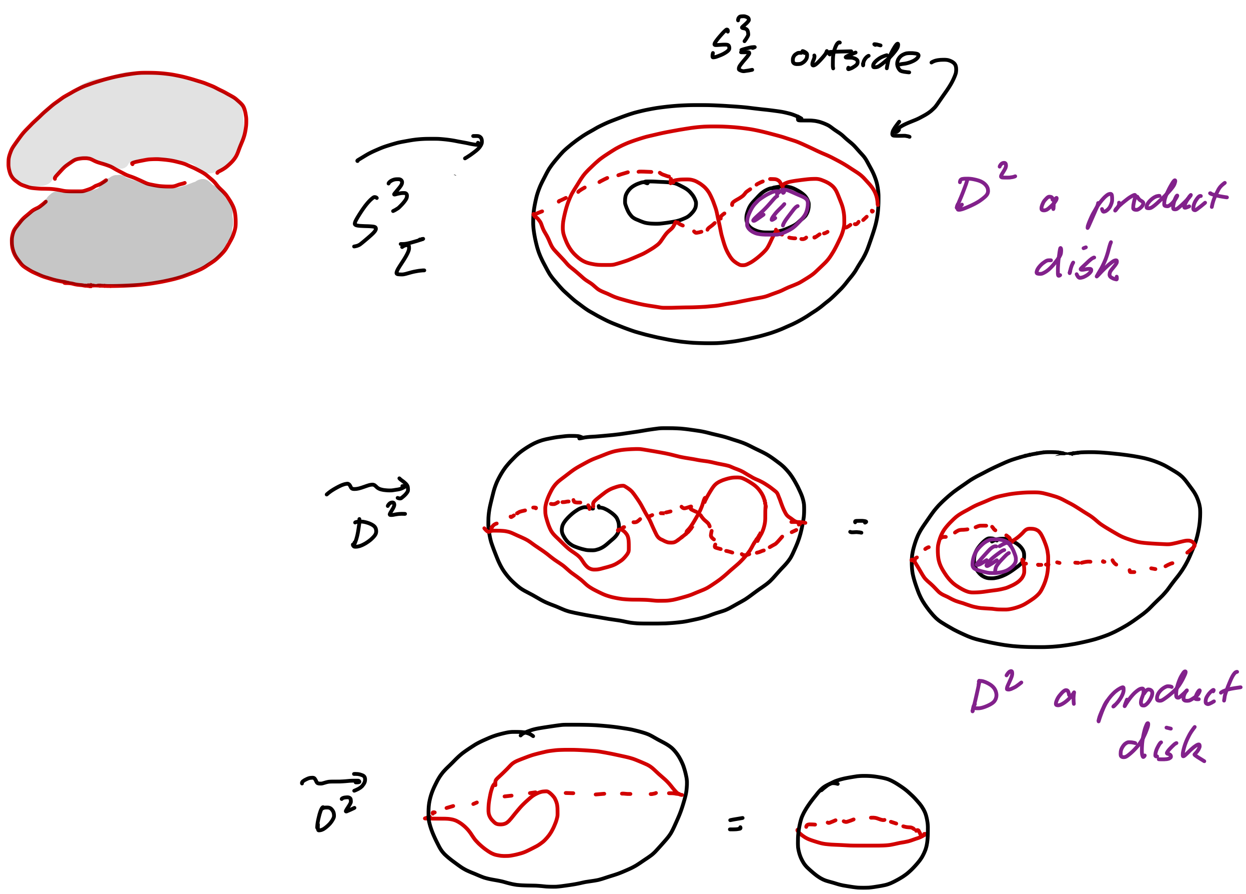
to do this we observe what happens when cutting on a product disk





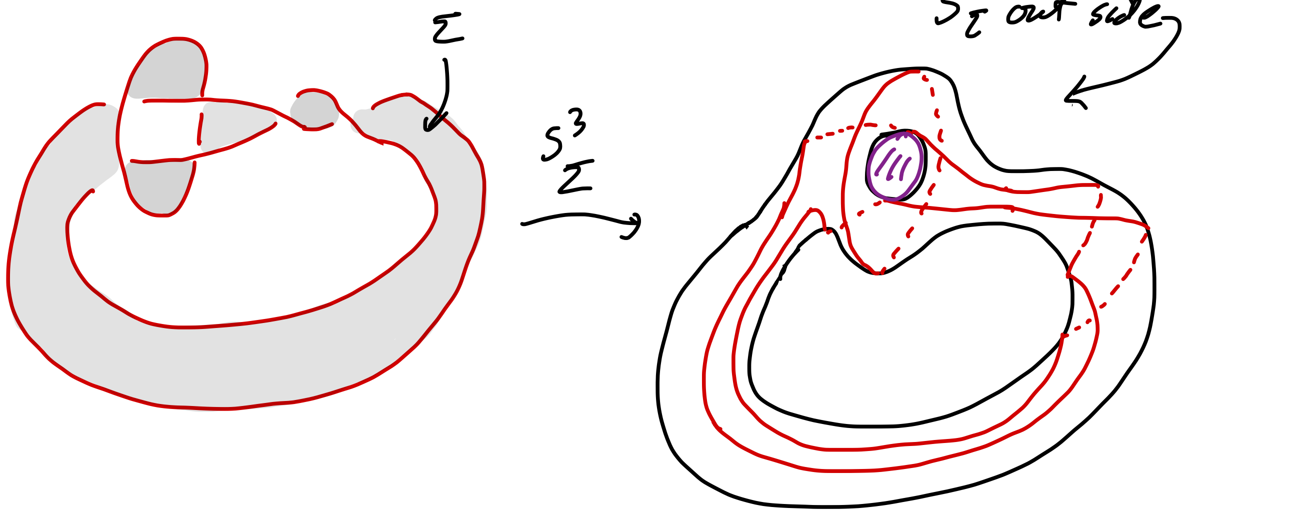
examples:

1)

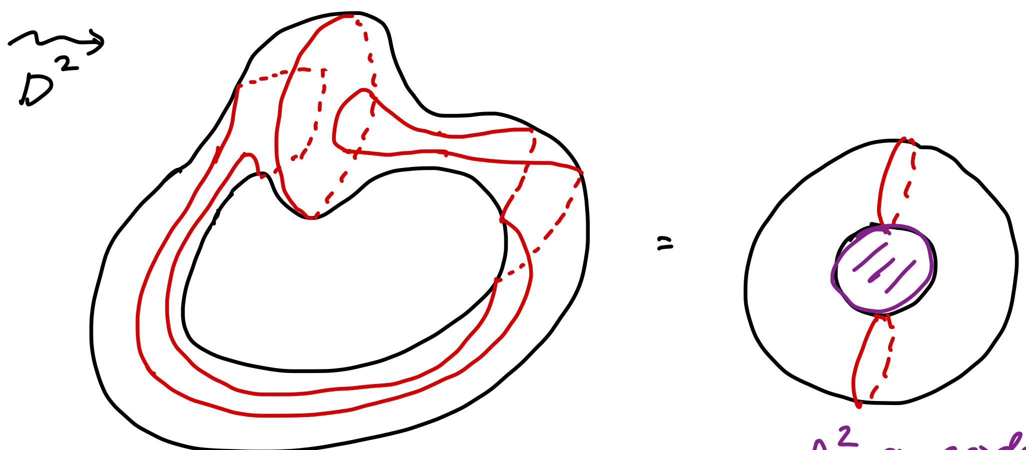


so  is fibered

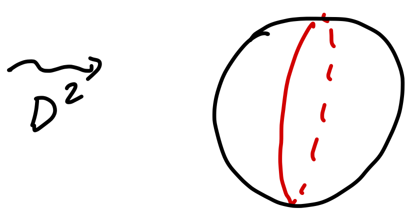
2)



D^2 a product disk



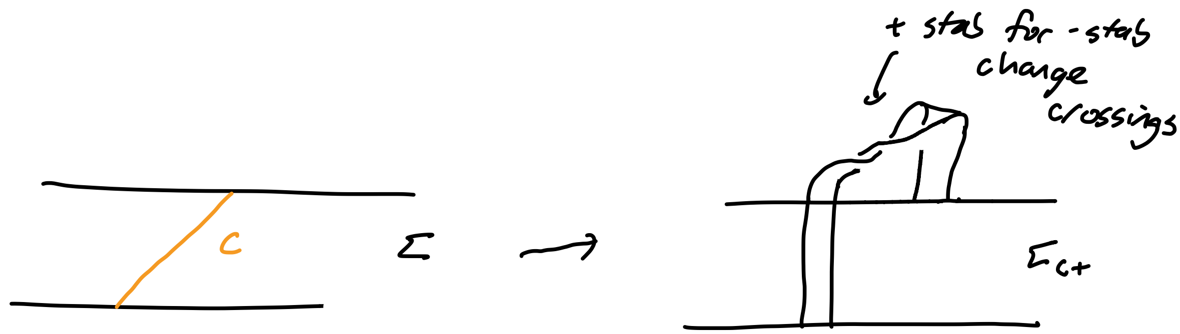
D^2 a product disk



so link fibered

if $K = \partial\Sigma$ in Y and $c \subset \Sigma$ a properly embedded arc the a $\pm$ stabilization of Σ along c

is

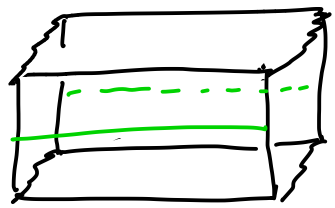


lemma 11:

$\boxed{\text{if } \partial\Sigma \text{ fibered then } \partial\Sigma_{c\pm} \text{ fibered}}$

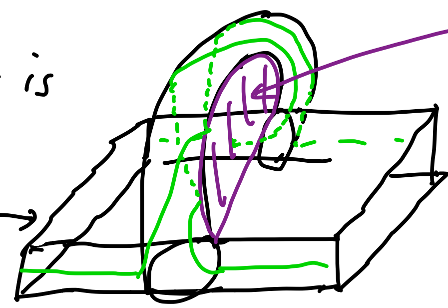
Proof:

a nbhd of Σ is



and its complement is disk decomposable

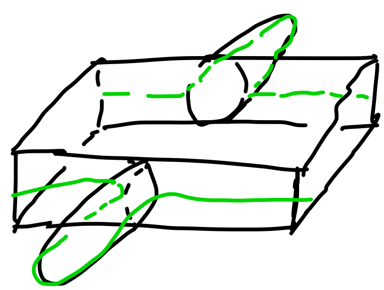
a nbhd of $\Sigma_{c\pm}$ is



product disk D^2

$Y_{\Sigma_{c\pm}}$ is complement of

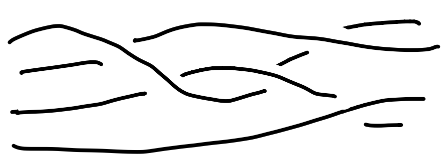
D^2



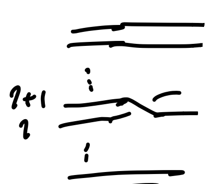
Y_Σ

now use disks form Y_Σ ∎

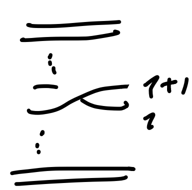
a braid is a bunch of strands moving from left to right



we let σ_i be



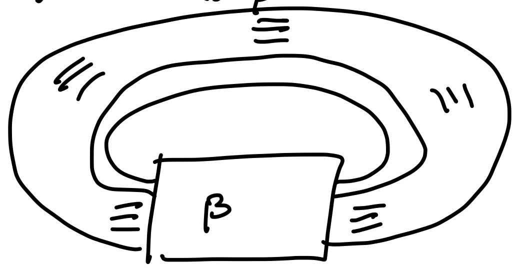
and σ_i^{-1} be



concatenate

any braid is a "product" of σ_i's

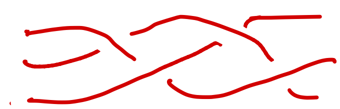
the closure of a braid β is

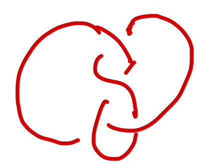


and is denoted $\hat{\beta}$

a braid is homogeneous if each σ_i occurs as σ_i or σ_i^{-1}
but you do not have σ_i and σ_i^{-1}

example:
$\beta = \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1^{-1}$



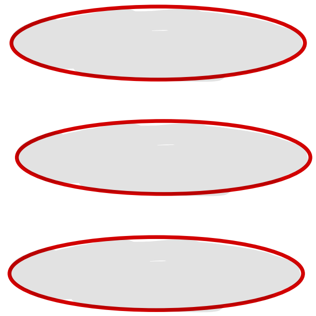
$\hat{\beta}$  $\cong$ 

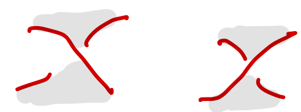
lemma 12: ← Stallings using different methods

$\boxed{\text{closures of homogeneous braids are fibered}}$

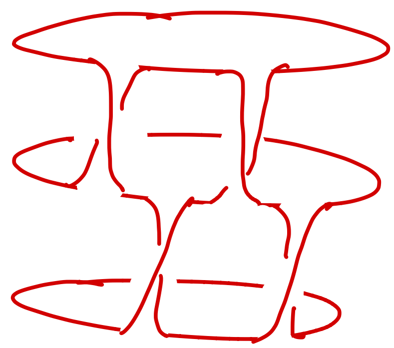
Proof: we first construct a nice Seifert surface for $\hat{\beta}$

for each strand in braid we take a disk



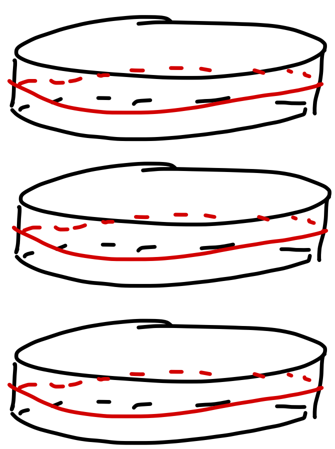
for each σ_1 or σ_1^{-1} add a band 

eg



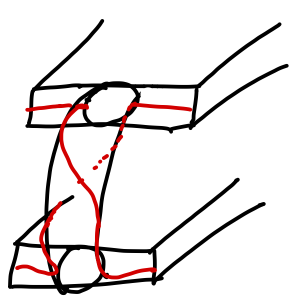
$\leftarrow$ genus 1 surface

now a nbhd of each disk is

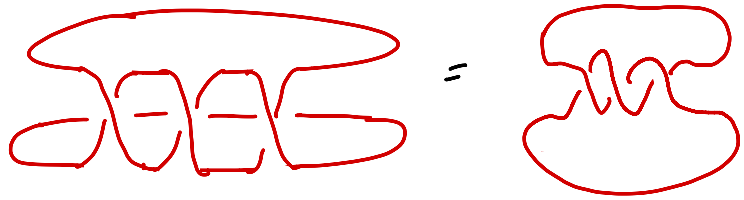


and each band adds

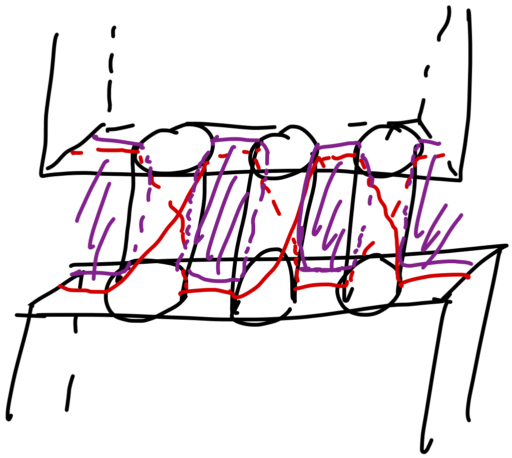
or reflect if σ_1^{-1}



now consider the region between i and $i+1$ sheet

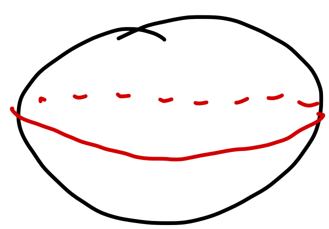


so a nbhd is



we get product disk

when we cut along disks the region between the i and $i+1$ disk becomes



so if we cut along all such disks we get a union of product B^3's $\therefore \hat{\beta}$ is fibered!